GUIDING CENTER MOTION

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ABSTRACT

The motion of charged particles in slowly varying electromagnetic fields is analyzed. The strength of the magnetic field is such that the gyro-period and the gyro-radius of the particle motion around field lines are the shortest time and length scales of the system. The particle motion is described as the sum of a fast gyro-motion and a slow drift velocity.

I. INTRODUCTION

The interparticle forces in ordinary gases are short-ranged, so that the constituent particles follow straight lines between collisions. At low densities where collisions become rare, the gas molecules bounce up and down between the walls of the containing vessel before experiencing a collision.

High-temperature plasmas, however, cannot be contained by a material vessel, but only by magnetic fields. The Lorentz forces that act on the particles tie them to the magnetic field and force them to follow the field lines. In order to confine the particles in a bounded volume, the magnetic field must be curved and inhomogeneous. In addition, it must be strong. So strong, that the Lorentz force dominates all other forces. Therefore, charged particles do not follow straight lines between collisions but follow strongly curved orbits under the influence of the magnetic field. In fact, many properties of a magnetically confined plasma are dominated by the motion of the particles subject to the Lorentz force $q\boldsymbol{v} \times \boldsymbol{B}$. Here $\boldsymbol{B}$ is the macroscopic field, i.e., the sum of externally applied field and the fields generated by the plasma particles collectively, but excluding the microscopic variations of the fields due to the individual particles.

The particle motion in the macroscopic field is the subject of this lecture. The microscopic fields, i.e., the interactions between individual particles ("collisions"), cause deviations from these particle orbits. Collisions in a plasma are caused by Coulomb interactions between the particles, with properties that are very different from collisions in a gas.

Firstly, the cross-section of Coulomb collisions is a strongly decreasing function of the energies of the interacting particles. Hence, the mean free paths of charged particles in high-temperature fusion devices are very long and the particles will trace out their trajectories over distances that can be comparable to or even larger than the size of the device before they are swept out of their orbits by collisions.

Secondly, the Coulomb force is a long range interaction. In a well-ionized plasma, particles rarely suffer large-angle deflections in two-particle collisions. Rather, their orbits are deflected through weak interactions with many particles simultaneously. Hence, the effects of collisions can be best described statistically, in terms of distributions of particles. The kinetic equation for the particle distribution function will be discussed at the end of this chapter, with emphasis on the role of the particle orbits, not the collisions.

The equations of motion of a particle with mass m and charge q in electromagnetic fields $\boldsymbol{E}(\boldsymbol{x},t)$ and $\boldsymbol{B}(\boldsymbol{x},t)$ are,

$$\dot{\boldsymbol{x}} = \boldsymbol{v}, \qquad \dot{\boldsymbol{v}} = \frac{q}{m}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}), \tag{1}$$

where the dot denotes the time derivative. Each of the N plasma particles satisfies such equations. The solutions to the $6N$ equations are the particle trajectories. These trajectories determine the local charge and current density which are the sources in Maxwell's equations and which determine the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$. In turn, these fields determine the particle trajectories. This self-consistent picture is extremely complex.

However, as illustrated above, in a weakly collisional plasma one can first study the behaviour of test particles in given fields $\boldsymbol{E}(\boldsymbol{x},t)$ and $\boldsymbol{B}(\boldsymbol{x},t)$. The role of the particles as sources of charge density and current in Maxwell's equations is disregarded. The fields $\boldsymbol{E}$ and $\boldsymbol{B}$ of course obey the subset of Maxwell's equations,

$$\nabla \times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t, \qquad \nabla \boldsymbol{\cdot} \boldsymbol{B} = 0. \tag{2}$$

II. GYRATION AND DRIFT

A. Motion in a Constant Magnetic Field

Let us first consider the motion of a charged particle in the presence of a constant magnetic field $\boldsymbol{B}$,

$$m\dot{\boldsymbol{v}} = q(\boldsymbol{v} \times \boldsymbol{B}).$$

The kinetic particle energy remains constant because the Lorentz force is always perpendicular to the velocity and can thus change only its direction, but not its magnitude. The particle velocity can be decomposed into components parallel and perpendicular to the magnetic field, $\boldsymbol{v} = v_{\parallel}\boldsymbol{b} + \boldsymbol{v}_{\perp}$, where $\boldsymbol{b} \equiv \boldsymbol{B}/B$ is the unit vector in the direction of

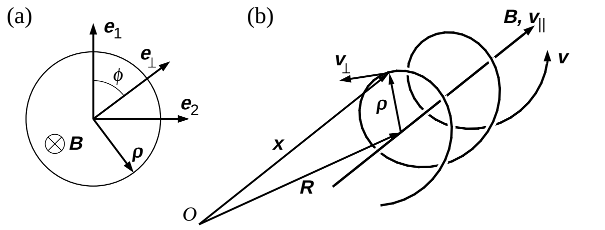


Figure 1: Definition of the gyro-angle ϕ (a) and guiding center (b).

$\boldsymbol{B}$. The Lorentz force does not affect the parallel motion: $v_\parallel$ = constant. Only $\boldsymbol{v}_\perp$ interacts with $\boldsymbol{B}$, leading to a circular motion perpendicular to $\boldsymbol{B}$. The centrifugal force $mv_\perp^2/r$ balances the Lorentz force $qv_\perp B$ for a gyration radius r equal to the "Larmor radius"

$$\rho \equiv \frac{mv_\perp}{|q|B}.$$

If we set $\frac{1}{2}mv_\perp^2 = kT$ for the two dimensional thermal motion $\perp$ $\boldsymbol{B}$, we obtain $\rho = (2mkT)^{1/2}/|q|B$. In a typical fusion plasma ($kT = 10\,\text{keV}$, $B = 5\,\text{T}$) the electrons have a gyroradius of 67 μm and deuterons 4.1 mm.

The frequency of the gyration, called cyclotron frequency ω_c, follows from $v_\perp = \omega_c\,\rho$,

$$\omega_c = qB/m.$$

In fusion experiments the electron cyclotron frequency is of the same order of magnitude as the plasma frequency. Alhough the particle motion in a constant field is elementary, the following notation will also serve more complicated cases. Let $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ be unit vectors perpendicular to each other and to $\boldsymbol{b}$, and define co-rotating unit vectors (Fig. 1(a)):

$$\begin{aligned}\boldsymbol{e}_\perp(t) &= \boldsymbol{e}_1 \cos\phi + \boldsymbol{e}_2 \sin\phi,\\ \boldsymbol{e}_\rho(t) &= \boldsymbol{e}_2 \cos\phi - \boldsymbol{e}_1 \sin\phi, \qquad \phi = \phi_0 - \omega_c t.\end{aligned}$$

As illustrated in Fig. 1(b), the particle position $\boldsymbol{x}$ can be decomposed into a **guiding center** position $\boldsymbol{R}$ that moves with velocity $v_\parallel \boldsymbol{b}$, and a rotating gyration radius vector $\boldsymbol{\rho}$,

$$\boldsymbol{x} = \boldsymbol{R} + \boldsymbol{\rho}, \tag{3a}$$

$$\boldsymbol{\rho} = -\frac{m}{qB^2}\boldsymbol{v}\times\boldsymbol{B} = \rho\,\text{sgn}(q)\,\boldsymbol{e}_\rho, \tag{3b}$$

$$\boldsymbol{v}_\perp = \dot{\boldsymbol{\rho}} = v_\perp \boldsymbol{e}_\perp. \tag{3c}$$

The particle trajectory is a helix around the guiding center magnetic field line (Fig. 2).

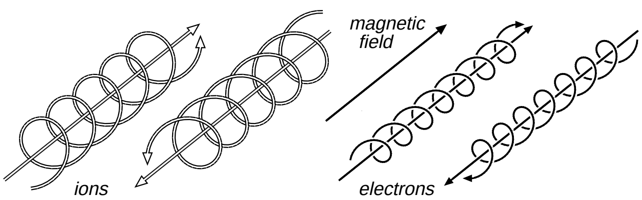
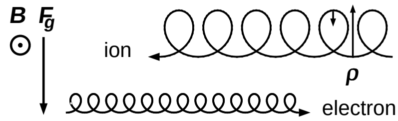


Figure 2: Orientation of the gyration orbits of electrons and ions in a magnetic field. The guiding center motion is also shown.

Figure 3: ion and electron drifts mg/qB in a gravitational field.



B. Drift due to an Additional Force

If, in addition to the Lorentz force, a constant force $\boldsymbol{F}$ acts on the charged particle, the equation of motion is

$$m\dot{\boldsymbol{v}} = q\,(\boldsymbol{v}\times\boldsymbol{B}) + \boldsymbol{F}. \tag{4}$$

The motion of the particle due to $\boldsymbol{F}$ can be separated from the gyration due to $\boldsymbol{B}$ by using the guiding center as reference frame. Again the guiding center position $\boldsymbol{R}$, the position of the particle $\boldsymbol{x}$, and the gyration radius vector $\boldsymbol{\rho}$ are related as in Eq. (3). The velocity of the guiding center can be obtained by differentiating the equation $\boldsymbol{R} = \boldsymbol{x} - \boldsymbol{\rho}$,

$$\begin{aligned}\boldsymbol{v}_g \equiv \dot{\boldsymbol{R}} &= \dot{\boldsymbol{x}} - \dot{\boldsymbol{\rho}}\\ &= \boldsymbol{v} + \frac{m}{qB^2}\,\dot{\boldsymbol{v}}\times\boldsymbol{B}\\ &= \boldsymbol{v} + \frac{1}{qB^2}\,(q\boldsymbol{v}\times\boldsymbol{B} + \boldsymbol{F})\times\boldsymbol{B}.\end{aligned}$$

Using $(\boldsymbol{v}\times\boldsymbol{B})\times\boldsymbol{B} = -\boldsymbol{v}_\perp B^2$ and $\boldsymbol{v} - \boldsymbol{v}_\perp = v_\parallel \boldsymbol{b}$ we obtain

$$\boldsymbol{v}_g = v_\parallel \boldsymbol{b} + \frac{\boldsymbol{F}\times\boldsymbol{B}}{qB^2}.$$

Thus, one sees that any force with a component perpendicular to $\boldsymbol{B}$ causes a particle to drift perpendicular to both $\boldsymbol{F}$ and $\boldsymbol{B}$. The basic mechanism for a drift in this direction is a periodic variation of the gyro-radius. When a particle accelerates in a force field, the gyroradius increases and when it slows down its gyroradius decreases, leading to the nonclosed trajectories shown in Fig. 3. The net effect is a drift perpendicular to the force and the magnetic field.

A force parallel to $\boldsymbol{B}$ does not lead to a drift, but simply causes a parallel acceleration as can be seen from Eq. (4). Summarizing,

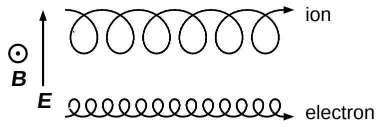
$$\boldsymbol{v}_{g,\perp} = \frac{\boldsymbol{F}_\perp\times\boldsymbol{B}}{qB^2}, \qquad \frac{dv_{g,\parallel}}{dt} = \frac{F_\parallel}{m}. \tag{5}$$

An example is the drift due to a constant gravitational force $F_g = mg$ perpendicular to the magnetic field. The resulting drift velocity, $v_g = mg/qB$, is in opposite directions for electrons and ions (see Fig. 3). The net effect is a current density. However, in laboratory plasmas v_g is far to small to be of importance (2×10^{-8}m/s in a magnetic field $B = 5\,\text{T}$).

C. $\boldsymbol{E}\times\boldsymbol{B}$ Drift

A different situation arises in the presence of a constant electric force $q\boldsymbol{E}$. Since the electric force is in opposite directions for electrons and ions, the resulting drift velocity,

Figure 4: $\boldsymbol{E}\times\boldsymbol{B}$ drift of ions and electrons.



$$\boldsymbol{v}_E = \frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2}, \tag{6}$$

does not depend on the sign of the charge or the particles. It is also independent of the particle mass and therefore identical for ions and electrons. Hence, this drift leads to a net flow of the plasma, not to a current.

D. Polarization Drift

If the electric field is spatially constant but depends on time, $\partial\boldsymbol{E}/\partial t \neq 0$, the $\boldsymbol{E}\times\boldsymbol{B}$ drift (6) is not constant. Instead, there is an acceleration $\perp \boldsymbol{B}$ which can be thought of as being caused by a force

$$\boldsymbol{F} = m\frac{d\boldsymbol{v}_E}{dt} = \frac{m}{B^2}\frac{\partial\boldsymbol{E}}{\partial t}\times\boldsymbol{B}.$$

This force, according to Eq. (5), yields yet another drift,

$$\boldsymbol{v}_p = \frac{\boldsymbol{F}\times\boldsymbol{B}}{qB^2} = \frac{m}{qB^2}\frac{\partial\boldsymbol{E}}{\partial t}.$$

This secondary drift is the polarization drift, which depends on the charge and the mass of the particle. The associated current density is

$$\boldsymbol{j}_p = \frac{\rho_m}{B^2}\frac{\partial\boldsymbol{E}}{\partial t},$$

where $\rho_m = m_e n_e + m_i n_i$ is the mass density. The electron contribution to this current density is a factor $\mathcal{O}(m_e/m_i)$ smaller than the contribution from the ions.

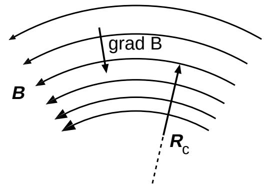
E. Particle Drift in Inhomogeneous Magnetic Fields

For spatially slowly varying magnetic fields, Eq. (5) can still be applied if the relative variation of $\boldsymbol{B}$ along one gyration of the particle is small.

One type of field inhomogeneity that gives rise to a drift is curvature of the magnetic field lines. For a particle that moves along a curved magnetic field line the separation of its velocity into $v_\perp$ and $v_\parallel$ changes with its position. This effect will be taken into account systematically in Section IV. In the present section we will give an intuitive argument that shows how field line curvature can cause drift motion. The curvature is given by $\nabla_\parallel\boldsymbol{b} = -\boldsymbol{R}_c/R_c^2$, a vector $\perp \boldsymbol{B}$. Here $\nabla_\parallel \equiv \boldsymbol{b}\cdot\nabla$ is the gradient along $\boldsymbol{B}$ and $\boldsymbol{R}_c$ is the curvature radius shown in Fig. 5. A particle which follows the curved field line with velocity $v_\parallel$ experiences a centrifugal force $\boldsymbol{F}_c = mv_\parallel^2\,\boldsymbol{R}_c/R_c^2$, which is responsible for the drift velocity

$$\boldsymbol{v}_c = \frac{mv_\parallel^2}{qB^2}\,\boldsymbol{B}\times\nabla_\parallel\boldsymbol{b}. \tag{7}$$

Figure 5: Inhomogeneous magnetic field. Relation between the curvature radius and the field gradient in a force-free magnetic field ($\nabla\times\boldsymbol{B}\parallel\boldsymbol{B}$).



The other inhomogeneity that results in a drift is the transverse gradient of the magnetic field strength. The particle orbit has a smaller radius of curvature on that part of its orbit located in the stronger magnetic field. This leads to a drift perpendicular to both the magnetic field and its gradient. The drift is not the result of a constant force, and hence Eq. (5) cannot be applied directly.

Instead we discuss the averaged effect of ∇B on the gyro-orbit by considering the current $I = q\omega_c/2\pi$ associated with the gyro-motion of a charged particle. The magnetic moment is defined as the product of the current and the area which is surrounded by the current. Since the area encompassed by the gyro-orbit equals $\pi\rho^2$, the magnetic moment per unit particle mass is

$$\mu = \pi\rho^2\frac{I}{m} = \pi\rho^2\frac{q^2B}{2\pi m^2} = \frac{v_\perp^2}{2B}. \tag{8}$$

The gyro-averaged force equals the force on a magnetic dipole in a magnetic field gradient,

$$\boldsymbol{F}_{\nabla B} = -m\mu\,\nabla B. \tag{9}$$

Application of Eq. (5) to this force yields the ∇B-drift,

$$\boldsymbol{v}_{\nabla B} = \frac{mv_\perp^2}{2qB^3}\,\boldsymbol{B}\times\nabla B. \tag{10}$$

The curvature and ∇B drifts are often comparable. In a plasma in equilibrium one has approximately $\nabla\times\boldsymbol{B}\parallel\boldsymbol{B}$. For a pressure gradient $\nabla p = 0$ this relation is exact. It implies a relation between the curvature vector and ∇B, illustrated in Fig 5,

$$\nabla_\parallel\boldsymbol{b} = \frac{\nabla_\perp B}{B}. \tag{11}$$

Using this relation, the ∇B and curvature drifts (10) and (7) can be combined to

$$\boldsymbol{v}_c + \boldsymbol{v}_{\nabla B} = \frac{m}{qB^3}(v_\parallel^2 + \tfrac{1}{2}v_\perp^2)\,\boldsymbol{B}\times\nabla B. \tag{12}$$

Averaged over a thermal velocity distribution, this drift velocity equals $2T/qBR_c = 2v_{th}\rho_{th}/R_c$.

As a first example of these drifts, consider the electrons and protons captured in the earth's magnetic field (trapping in a magnetic field will be discussed in the next section). Due to the gradient and curvature of the earth's magnetic field, the electrons and protons captured in this field drift around the equator, the electrons from west to east and the protons in the opposite direction, producing the so-called 'electron current' shown in Fig 6.

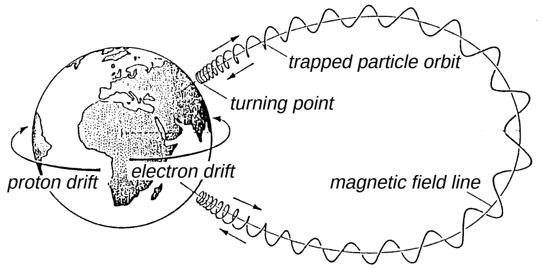


Figure 6: Electron and proton drifts in the Earth magnetic field.

F. Plasma Diamagnetism

The current of a gyrating particle generates a magnetic field in the direction opposite to the given field $\boldsymbol{B}$, so that a plasma is diamagnetic. The contributions to the current density of neighbouring gyrating particles cancel each other in a homogeneous plasma. The magnetization of the medium is found by summing over all particles, $\boldsymbol{M} = -n\langle m\mu\rangle\boldsymbol{b}$. In a thermal plasma $\langle \frac{1}{2}mv_\perp^2\rangle = T$ and therefore $\langle m\mu\rangle = T/B$ and $\boldsymbol{M} = -\boldsymbol{b}p/B$. Here n is the particle density and p the pressure. If the pressure is not constant, the magnetization causes a **diamagnetic current**

$$\boldsymbol{j}_D = \nabla \times \boldsymbol{M} = -\frac{\nabla p \times \boldsymbol{B}}{B^2}.$$

This current precisely agrees with the force balance in a conducting fluid, $\nabla p = \boldsymbol{j} \times \boldsymbol{B}$. Here, the force per unit volume $\boldsymbol{j}\times\boldsymbol{B}$ is the Lorentz force $q\boldsymbol{v}\times\boldsymbol{B}$ summed over all particles, making use of $\boldsymbol{j} = \sum nq\boldsymbol{v}$. If one views the electrons and ions in the plasma as separate fluids, the diamagnetism is found to give different contributions to the ion and electron fluid velocities, the **diamagnetic velocities**,

$$\boldsymbol{v}_{D,i} = -\frac{\nabla p_i \times \boldsymbol{B}}{q_i n B^2}, \qquad \boldsymbol{v}_{D,e} = \frac{\nabla p_e \times \boldsymbol{B}}{enB^2},$$

which resemble drift velocities of the form (5). Their relation to the diamagnetic current is $\boldsymbol{j}_D = n_i q_i \boldsymbol{v}_{D,i} - n_e e \boldsymbol{v}_{D,e}$.

III. ADIABATIC INVARIANTS

When a system performs a periodic motion, the action integral $I = \oint P\,dQ$, taken over one period, is a constant of motion, where P is a generalized momentum and Q the corresponding coordinate. For slow changes of the system (compared with the characteristic time of the periodic motion) the integral I remains constant and is called an 'adiabatic invariant'. More precisely: if the system changes on a timescale τ, and the frequency of the periodic motion is ω, then changes to I of the order $\Delta I \sim e^{-\omega\tau}$ can be expected.

A. Magnetic Moment

The first adiabatic invariant is the magnetic moment $\mu = v_\perp^2/2B$ defined in Eq. (8), which is proportional to the magnetic flux $\pi\rho^2 B$ enclosed by the gyro-orbit. The periodic motion is the Larmor gyration, P is the angular momentum $mv_\perp\rho$ and the coordinate Q is the angle ϕ. We get

$$\oint P\,dQ = \oint mv_\perp\rho\,d\phi = 2\pi\rho m v_\perp = 4\pi\,\frac{m^2}{q}\,\mu.$$

Note that μ is no longer a constant of motion if the charge q changes, for instance due to ionization or charge exchange, which preferentially occurs at the edge of the plasma.

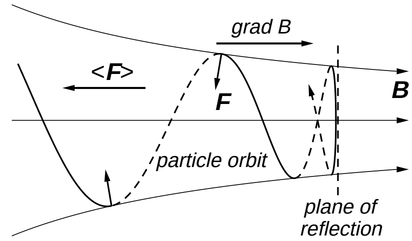


Figure 7: Reflected particle orbit in a magnetic mirror.

B. Particle Trapping

The invariance of μ plays a role in magnetic mirrors. The mirror effect occurs when a particle guiding center moves towards a region with a stronger magnetic field. As Fig. 7 shows, field lines encountered by the particle gyro-orbit converge. Hence the Lorentz force has a gyro-averaged component opposite to ∇B. This mirror force $\parallel \boldsymbol{B}$ is precisely the force on a magnetic dipole of strength μ in a gradient $\nabla_\parallel B$, given by Eq. (9). According to Eq. (5) this force causes a parallel deceleration

$$\dot{v}_\parallel = -\mu\nabla_\parallel B. \tag{13}$$

Since the particle experiences a magnetic field change $\dot{B} = v_\parallel\nabla_\parallel B$, Eq. (13) and the conservation of energy $\epsilon = \frac{1}{2}v_\perp^2 + \frac{1}{2}v_\parallel^2 = \mu B + \frac{1}{2}v_\parallel^2$ imply that the magnetic moment μ is constant. In general, the change of the parallel velocity of a particle in a (spatially or temporally) varying magnetic field can be determined from the constancy of μ and ϵ in

$$v_\parallel(B) = \pm\sqrt{2(\epsilon - \mu B)}.$$

Figure 8 shows the principle of particle confinement in a mirror machine. The criterion for particle reflection ($v_\parallel = 0$) at the high field ends of the mirror machine is

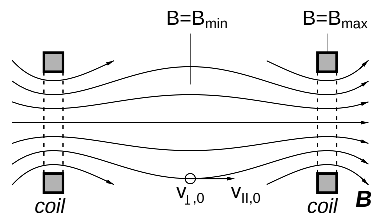


Figure 8: Magnetic field lines of a simple axisymmetric magnetic mirror.

$$\epsilon = \tfrac{1}{2}v_{\parallel,0}^2 + \mu B_{\min} \;\le\; \mu B_{\max}, \tag{14}$$

where $v_{\parallel,0}$ is the parallel velocity in the low field region. If we divide equation (14) by $\mu B_{\min} = \frac{1}{2}v_{\perp,0}^2$, we obtain

$$\frac{v_{\parallel,0}}{v_{\perp,0}} > \sqrt{B_{\max}/B_{\min} - 1} \tag{15}$$

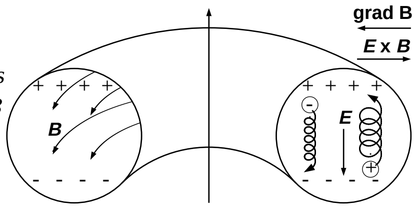


Figure 9: Vertical drifts and associated $\boldsymbol{E} \times \boldsymbol{B}$ drift in a toroidal field.

as the criterion for particle loss. In laboratory plasmas the mirror principle yields too large plasma losses at the open ends to be a promising candidate for fusion reactors. Coulomb collisions and certain instabilities cause a continuous transfer of trapped particles into the loss region (15).

The earth's magnetic field is also an example of a magnetic mirror. It forms two belts of confined charged particles originating from the solar wind (see Fig. 6).

A second adiabatic invariant, the longitudinal invariant $J = \oint m v_\parallel \, d\ell$, is defined as the integral over the periodic orbit for trapped particles in mirror geometries. Defining the length L between two turning points and the average longitudinal velocity $\langle v_\parallel \rangle$, the constant of motion is $J = 2m\langle v_\parallel \rangle L$. When L decreases, $\langle v_\parallel \rangle$ increases. This is the basis of the Fermi acceleration principle of cosmic radiation.

C. Toroidal Systems: the Tokamak

The end losses inherent to mirror devices are avoided in the closed geometry of toroidal systems. It is important to realize that in a simple toroidal magnetic field (Fig. 9), the magnetic field curvature and gradient (Fig. 5) give rise to **vertical drifts** that are in opposite directions for ions and electrons. The resulting charge separation causes an outward $\boldsymbol{E} \times \boldsymbol{B}$ drift for electrons and ions alike. A plasma in a toroidal field alone will thus be unstable.

This conclusion can also be reached by considering the Lorentz force $\boldsymbol{j} \times \boldsymbol{B}$ on the plasma as a whole instead of the individual particle orbits. With the current density given by $\boldsymbol{j} = \nabla \times \boldsymbol{B}/\mu_0$, it can be shown that this force cannot "point inward" everywhere to confine a plasma in a purely toroidal field.

Therefore, in toroidal plasma devices additional magnetic field components are required in order to reach a steady state where the plasma pressure is balanced by magnetic forces ($\nabla p = \boldsymbol{j} \times \boldsymbol{B}$). The required twisted magnetic field is produced in tokamaks by the toroidal plasma current. As a consequence, particles approximately move on closed toroidal surfaces labelled by the poloidal magnetic flux ψ.

The vertical drifts average to zero over one poloidally closed particle orbit, as can be seen as follows. Because of the toroidal symmetry of $\boldsymbol{B}$, the canonical angular momentum associated to the toroidal angle is conserved exactly,

$$\begin{aligned} P_{\text{tor}} &= (m v_{\text{tor}} - q A_{\text{tor}}) R \\ &= m R v_{\text{tor}} - q\psi \\ &\simeq m R v_\parallel - q\psi = \text{constant}. \end{aligned} \tag{16}$$

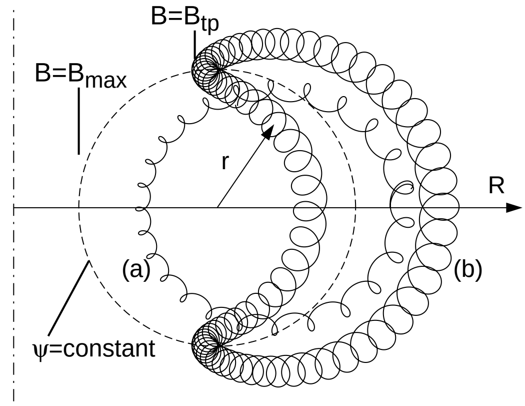


Figure 10: Projection of circulating (a) and trapped (b) particle orbits on the poloidal plane. R is the distance to the vertical axis.

Hence, because $v_\parallel$ remains in the range determined by ϵ and μ, the particle remains in a bounded ψ zone and does not escape in the vertical direction.

In a tokamak, the field strength has its maximum value at the inside of the torus. A particle travelling along a field line feels a periodic mirror force. If the energy and magnetic moment of this particle have values such that $\epsilon > \mu B_{\text{max}}$, the particle is not reflected but continues its course and encircles the torus. These are circulating or transit particles.

On the other hand, if $\epsilon < \mu B_{\text{max}}$ the particle is reflected at the point where $\epsilon = \mu B_{\text{tp}}$ (see Fig. 10). The particle is trapped between magnetic mirrors and bounces between turning points. Thus, in leading order the particle executes a periodic motion along a field line.

The topology of the trajectories of trapped and circulating particles are quite different. While transit particles encircle the torus in the toroidal as well as in the poloidal direction, trapped particles may encircle the torus in the toroidal direction but be poloidally confined to the low field side of the torus. Due to this difference in topology, trapped and circulating particles often behave as different species.

Equation (16) shows that, because $v_\parallel$ of a trapped particle changes sign, its orbit is more strongly affected by the vertical drift than is a transit particle orbit. The projection of a trapped particle orbit on the poloidal plane of an axisymmetric torus is sketched in Fig. 10. The flux surfaces are assumed to have circular cross-sections. The width of this orbit can easily be calculated from Eq. (16). It follows that the total width, $\Delta r = \Delta\psi/(\partial\psi/\partial r)$, of the orbit is

$$\Delta r = 2\frac{m R v_\parallel}{q\, \partial\psi/\partial r} = 2\frac{v_{\parallel m}}{q B_{p,m}/m}, \tag{17}$$

where r is the cylindrical radius and $v_{\parallel m}$ is the value of the parallel velocity at the midplane. Note that the denominator in (17) is the gyro-frequency in the poloidal field at the midplane $B_{p,m}$. More about the particle orbits in tokamaks can be found in Refs. [1,2,3].

IV. GUIDING CENTER THEORY

A. Slowly Varying Fields

In the previous section, several aspects of charged particle orbits were presented separately under the assumption of either a constant magnetic field (gyration, $\boldsymbol{E}\times\boldsymbol{B}$ drift) or varying fields (∇B and curvature drifts, trapping).

This section presents a systematic ordering scheme that includes all components of the particle motion (gyration and drifts). This ordering scheme allows for **slow** variations in space and time of the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields. By this we mean that the characteristic length ℓ and time τ over which the fields vary appreciably are supposed to be large compared to the gyroradius ρ and the gyro-period ω_c^{-1}, respectively,

$$\frac{\rho}{\ell}=\frac{v_\perp}{\omega_c\ell}\ll 1, \qquad \frac{1}{\omega_c\tau}\ll 1.$$

Since ω_c^{-1} is proportional to m/q we may adopt $\delta\equiv m/q$ as a smallness parameter ($\rho\sim\delta\ell$, $\tau^{-1}\sim\delta\omega_c$) and expand Eq. (1) in powers of δ in order to obtain useful approximations to the particle motion. The aim of guiding center theory is to describe the particle motion as a fast gyro-motion around a slowly drifting guiding center.

Following Section II, we first perform a transformation from the phase-space variables $(\boldsymbol{x},\boldsymbol{v})$ to variables $(\boldsymbol{R},\boldsymbol{u})$ that are more suited to our purpose. We introduce the instantaneous position of the guiding center,

$$\boldsymbol{R}(t)=\boldsymbol{x}(t)-\boldsymbol{\rho}(t), \tag{18}$$

where

$$\boldsymbol{\rho}=\frac{\delta}{B}\,\boldsymbol{b}\times\boldsymbol{u}, \qquad \text{with} \qquad \boldsymbol{u}=\boldsymbol{v}-\boldsymbol{v}_E,$$

is the position of the particle with respect to this center. The cross field velocity $\boldsymbol{v}_E$ is defined by Eq. (6). The fields $\boldsymbol{b}$ and $\boldsymbol{v}_E$, and the field strength B are functions of space and time taken at the particle position $\boldsymbol{x}$ at time t. The instantaneous gyro-frequency is $\omega_c=B(\boldsymbol{x},t)/\delta$. It is convenient to introduce cylindrical coordinates $(v_\parallel,u_\perp,\phi)$ in $\boldsymbol{u}$-space,

$$\boldsymbol{u}=v_\parallel\boldsymbol{b}+u_\perp\boldsymbol{e}_\perp, \qquad \boldsymbol{e}_\perp=\boldsymbol{e}_1\cos\phi+\boldsymbol{e}_2\sin\phi. \tag{19}$$

The local, orthogonal unit vectors $(\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{b})$ are functions of $(\boldsymbol{x},t)$. From the transformation equations (18–19) and from the equation of motion (1) we obtain

$$\dot{\boldsymbol{R}}=v_\parallel\boldsymbol{b}+\boldsymbol{v}_E+\delta\boldsymbol{u}\times\frac{d}{dt}\frac{\boldsymbol{b}}{B}+\frac{\delta}{B}\boldsymbol{b}\times\frac{d\boldsymbol{v}_E}{dt}, \tag{20a}$$

$$\dot{v}_\parallel=\frac{1}{\delta}E_\parallel+(\boldsymbol{v}_E+u_\perp\boldsymbol{e}_\perp)\boldsymbol{\cdot}\frac{d\boldsymbol{b}}{dt}, \tag{20b}$$

$$\dot{u}_\perp=-\boldsymbol{e}_\perp\boldsymbol{\cdot}\big(v_\parallel\frac{d\boldsymbol{b}}{dt}+\frac{d\boldsymbol{v}_E}{dt}\big), \tag{20c}$$

$$\dot{\phi}=-\frac{B}{\delta}-\boldsymbol{e}_2\boldsymbol{\cdot}\frac{d\boldsymbol{e}_1}{dt}-\frac{1}{u_\perp}\boldsymbol{e}_\rho\boldsymbol{\cdot}\big(v_\parallel\frac{d\boldsymbol{b}}{dt}+\frac{d\boldsymbol{v}_E}{dt}\big), \tag{20d}$$

In order to obtain a closed set of guiding center equations of motion, the quantities on the RHS of Eqs. (20), which are given in phase space variables $(\boldsymbol{x},\boldsymbol{v})$, must be expressed in terms of the guiding center variables $(\boldsymbol{R},v_\parallel,u_\perp,\phi)$. This we can do only approximately. Firstly, the fields $\boldsymbol{E}$, $\boldsymbol{B}$ (and the related vectors $\boldsymbol{b}$, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{v}_E$) at the particle position $\boldsymbol{x}$ are expanded in a Taylor series around the guiding center position $\boldsymbol{R}$, e.g.,

$$\begin{aligned}\boldsymbol{b}(\boldsymbol{x},t)&=\boldsymbol{b}(\boldsymbol{R},t)+\boldsymbol{\rho}\boldsymbol{\cdot}\nabla\boldsymbol{b}(\boldsymbol{R},t)+\dots\\&=\boldsymbol{b}(\boldsymbol{R},t)+\delta\frac{u_\perp}{B}\boldsymbol{e}_\rho\boldsymbol{\cdot}\nabla\boldsymbol{b}(\boldsymbol{R},t)+\mathcal{O}(\delta^2).\end{aligned} \tag{21}$$

Secondly, we note that the total time derivatives on the RHS of Eqs. (20) are derivatives along the particle trajectory,

$$\frac{d}{dt}=\frac{\partial}{\partial t}+(v_\parallel\boldsymbol{b}+\boldsymbol{v}_E+u_\perp\boldsymbol{e}_\perp)\boldsymbol{\cdot}\nabla.$$

In terms of the guiding center variables this becomes

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\dot{\boldsymbol{R}}\boldsymbol{\cdot}\frac{\partial}{\partial\boldsymbol{R}}+\dot{v}_\parallel\frac{\partial}{\partial v_\parallel}+\dot{u}_\perp\frac{\partial}{\partial u_\perp}+\dot{\phi}\frac{\partial}{\partial\phi}. \tag{22}$$

In view of Eq. (21). this procedure leads to an approximate set of equations. Nevertheless, we have made an important gain. While in the original equation (1) all forces are $\mathcal{O}(\delta^{-1})$, in Eqs. (20) this order occurs only on the right of Eqs. (20b, d).

The term in Eq. (20b) that contains the parallel electric field $E_\parallel=\boldsymbol{E}\cdot\boldsymbol{b}$ tends to be large, $\mathcal{O}(\delta^{-1})$. If this would actually be the case, then particles would be accelerated on the timescale of the gyro-rotation. Since ions and electrons are accelerated in opposite directions, this would lead to charge separation and, thus, to the generation of electric fields on the shortest timescale. This would violate our requirement of slowly varying fields. Therefore, we must require that the parallel electric field is small, $E_\parallel=\mathcal{O}(\delta)$. Hence, the first term in the RHS of Eq. (20d) is the only $\mathcal{O}(\delta^{-1})$ term, and the gyro-angle ϕ is the only fast varying quantity. All other quantities vary on the timescale $\mathcal{O}(\delta^0)$ of the fields.

B. Guiding Center Motion

Neglecting all space-time dependences of the fields one immediately reobtains from Eqs. (20) the results of Section II,

$$\begin{aligned}u_\perp&=\text{constant}, \qquad &\phi&=-\omega_c t+\phi_0,\\ v_\parallel&=\text{constant}, \qquad &d\boldsymbol{R}/dt&=v_\parallel\boldsymbol{b}+\boldsymbol{v}_E.\end{aligned} \tag{23}$$

Since we are interested in the motion on the timescale on which the fields vary we have to retain next order contributions to the motion (23). This means that we have to retain $\mathcal{O}(\delta^0)$ contributions in Eqs. (20b, c, d) and $\mathcal{O}(\delta)$ contributions in Eq. (20a).

By applying the expansion (21) in the RHS of Eqs. (20) and neglecting contributions that are higher order in δ, one obtains a set of equations for the variables $z\in\{\boldsymbol{R},v_\parallel,u_\perp,\phi\}$ of the form

$$\begin{aligned}\frac{dz}{dt}&=f_z(\boldsymbol{R},v_\parallel,u_\perp,\phi)\\&=\langle f_z\rangle+\tilde{f}_z,\end{aligned} \tag{24}$$

where $\langle\cdots\rangle = (2\pi)^{-1}\oint d\phi\,(\cdots)$ denotes the average over the gyro-angle ϕ. The function $\tilde{f}_z$ is periodic in the fast-oscillating gyro-angle ϕ. The oscillatory components $\tilde{f}_z$ can be removed to leading order by means of a small, $\mathcal{O}(\delta)$, redefinition of the variables z,

$$\bar{z} = z + \frac{\delta}{B}\int^{\phi}\tilde{f}_z(\phi')d\phi'. \tag{25}$$

From Eq. (20d) one sees that $d\phi/dt = -B/\delta + \mathcal{O}(\delta^0)$. Therefore in Eq. (22) the last term dominates,

$$\frac{d}{dt} = -\frac{B}{\delta}\frac{\partial}{\partial\phi} + \mathcal{O}(\delta^0),$$

so that the time derivative of Eq. (25) yields, as intended,

$$\frac{d\bar{z}}{dt} = \langle f_z\rangle(\bar{\boldsymbol{R}}, \bar{v}_\parallel, \bar{u}_\perp, \bar{\phi}) + \mathcal{O}(\delta). \tag{26}$$

Dropping the bar notation, the explicit forms of Eq. (26) are

$$\dot{\boldsymbol{R}} = (v_\parallel + \delta\frac{u_\perp^2}{2B}\boldsymbol{b}\boldsymbol{\cdot}\nabla\times\boldsymbol{b})\boldsymbol{b} + \boldsymbol{v}_E + \delta\frac{u_\perp^2}{2B}\boldsymbol{b}\times\frac{\nabla B}{B} + \delta\frac{\boldsymbol{b}}{B}\times D_t(v_\parallel\boldsymbol{b} + \boldsymbol{v}_E) + \mathcal{O}(\delta^2), \tag{27a}$$

$$\dot{v}_\parallel = \frac{E_\parallel}{\delta} - \frac{u_\perp^2}{2B}\nabla_\parallel B + \boldsymbol{v}_E\boldsymbol{\cdot}D_t\boldsymbol{b} + \mathcal{O}(\delta), \tag{27b}$$

$$\dot{u}_\perp = \frac{v_\parallel u_\perp}{2B}\nabla_\parallel B - \frac{u_\perp}{2}(\nabla\boldsymbol{\cdot}\boldsymbol{v}_E - \boldsymbol{b}\boldsymbol{\cdot}\nabla_\parallel\boldsymbol{v}_E) + \mathcal{O}(\delta), \tag{27c}$$

$$\dot{\phi} = -\frac{B}{\delta} - \boldsymbol{e}_2\boldsymbol{\cdot}D_t\boldsymbol{e}_1 - \frac{v_\parallel}{2}\boldsymbol{b}\boldsymbol{\cdot}\nabla\times(v_\parallel\boldsymbol{b} + \boldsymbol{v}_E) + \mathcal{O}(\delta), \tag{27d}$$

where all fields are taken at the guiding center position $\boldsymbol{R}$, and where to leading order

$$D_t \equiv \frac{\partial}{\partial t} + (v_\parallel\boldsymbol{b} + \boldsymbol{v}_E)\boldsymbol{\cdot}\nabla,$$

is the time derivative along the guiding center orbit.

Equation (27a) contains all drift effects in changing and inhomogeneous fields presented before. The second term on the RHS is the ∇B-drift given in Eq. (10). The last term, caused by changes of $v_\parallel\boldsymbol{b} + \boldsymbol{v}_E$ along the guiding center orbit, generalizes the curvature drift, given in Eq. (7). It also contains the polarization drift due to $d\boldsymbol{v}_E/dt$. The first term on the RHS includes an $\mathcal{O}(\delta)$ correction to the parallel velocity $v_\parallel$ proportional to the parallel current density: $\boldsymbol{b}\boldsymbol{\cdot}\nabla\times\boldsymbol{b} = \mu_0 j_\parallel/B$. More extensive treatments of the motion described by Eqs. (27) can be found in Refs. [4,5].

C. Magnetic Moment

Using Maxwell's equation (2) and the definition (6) of the $\boldsymbol{E}\times\boldsymbol{B}$-drift, the last term on the right of Eq. (27c) can be written as

$$\nabla\boldsymbol{\cdot}\boldsymbol{v}_E - \boldsymbol{b}\boldsymbol{\cdot}\nabla_\parallel\boldsymbol{v}_E = -\frac{1}{B}\frac{\partial B}{\partial t} - \frac{E_\parallel}{B}\boldsymbol{b}\boldsymbol{\cdot}\nabla\times\boldsymbol{b} - \boldsymbol{v}_E\boldsymbol{\cdot}\frac{\nabla B}{B}.$$

Recalling that $E_\parallel = \mathcal{O}(\delta)$, we obtain Eq. (27c) in the form

$$\dot{u}_\perp = \frac{u_\perp}{2B}D_t B + \mathcal{O}(\delta).$$

In terms of the magnetic moment per unit mass (see Eq. (8)),

$$\mu \equiv u_\perp^2/2B(\boldsymbol{R}, t),$$

this becomes

$$\dot{\mu} = \mathcal{O}(\delta).$$

It follows that **the magnetic moment is constant on the time and length scales of the field variations.**

The approximate constancy of μ is related with the fact that the gyro-angle ϕ is approximately an ignorable coordinate. From now on we shall use the variables $(\boldsymbol{R}, v_\parallel, \mu, \phi)$ instead of $(\boldsymbol{R}, v_\parallel, u_\perp, \phi)$

D. Kinetic Equation for Guiding Centers

The kinetic equation for the single-particle distribution function $f(\boldsymbol{x}, \boldsymbol{v}, t)$ is

$$\frac{df}{dt} = \Big(\frac{\partial}{\partial t} + \dot{\boldsymbol{x}}\boldsymbol{\cdot}\frac{\partial}{\partial\boldsymbol{x}} + \dot{\boldsymbol{v}}\boldsymbol{\cdot}\frac{\partial}{\partial\boldsymbol{v}}\Big)f = C(f), \tag{28}$$

where $C(f)$ is a collision operator. The coefficients $\dot{\boldsymbol{x}}$ and $\dot{\boldsymbol{v}}$ are given by Eq. (1). In terms of the new phase space variables $(\boldsymbol{R}, v_\parallel, \mu, \phi)$, the distribution function is

$$f(\boldsymbol{x}, \boldsymbol{v}, t) = F(\boldsymbol{R}, v_\parallel, \mu, \phi, t),$$

and the kinetic equation for F is

$$\Big(\frac{\partial}{\partial t} + \dot{\boldsymbol{R}}\boldsymbol{\cdot}\frac{\partial}{\partial\boldsymbol{R}} + \dot{v}_\parallel\frac{\partial}{\partial v_\parallel} + \dot{\mu}\frac{\partial}{\partial\mu} + \dot{\phi}\frac{\partial}{\partial\phi}\Big)F = C(F). \tag{29}$$

When all $\mathcal{O}(\delta)$ and higher order terms are neglected, the coefficients in Eq. (29) are given by Eqs. (27). These coefficients are independent of the gyro-angle ϕ. Upon averaging Eq. (29) over ϕ and substituting Eqs. (27), one obtains [6] the kinetic equation for the guiding center distribution function $\bar{F}(\boldsymbol{R}, v_\parallel, \mu, t) \equiv \langle F\rangle$,

$$\Big[D_t + \big(\frac{E_\parallel}{\delta} - \mu\nabla_\parallel B + \boldsymbol{v}_E\boldsymbol{\cdot}D_t\boldsymbol{b}\big)\frac{\partial}{\partial v_\parallel}\Big]\bar{F} = \langle C(F)\rangle. \tag{30}$$

V. DRIFT ORDERING

The leading order guiding center theory, as presented in the preceding Section, leads to a kinetic equation that governs the time evolution of the distribution function if the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields vary on the time scale set by the motion along field lines and by the cross-field drift motion,

$$\tau \sim \frac{\ell_\parallel}{v_\parallel} \sim \frac{\ell_\perp}{v_E} = \mathcal{O}(\delta^0),$$

where $\ell_\parallel, \ell_\perp$ are the characteristic lengths parallel and perpendicular to the magnetic field, respectively.

When the cross-field drift $\boldsymbol{v}_E$ is small and comparable to the magnetic drift velocities in Eq. (27a), $v_E \sim v_B = \mathcal{O}(\delta)$, the fields will vary on the longer timescale τ set by the drift motion around the system,

$$\frac{\ell_\parallel}{v_\parallel} \ll \tau \sim \frac{\ell_\perp}{v_B}.$$

This implies that $\partial/\partial t$, $\boldsymbol{v}_E\boldsymbol{\cdot}\nabla = \mathcal{O}(\delta)$ in Eq. (30). This is the **drift-ordering**. In order to obtain a kinetic equation that describes the evolution of the distribution function on this time scale, $\mathcal{O}(\delta)$ terms have to be retained in the coefficients of Eq. (30) and, thus, in the guiding center equations (27).

Higher order guiding center equations can be obtained in several ways. A particularly elegant method is given in Ref. [7], where a variational approach to guiding center theory, starting from the single particle Lagrangian in phase space, is given. Higher order equations can also be obtained by extending the method sketched in Eqs. (24–26). Since these methods are too elaborate for this introductory lecture, we will give only the results.

The new phase space variables $\bar{z}$ are defined by

$$\bar{\boldsymbol{R}} = \boldsymbol{x} - \boldsymbol{\rho} + \mathcal{O}(\delta^2), \qquad (31a)$$

$$\bar{v}_\parallel = v_\parallel + \delta\mu\boldsymbol{b}\boldsymbol{\cdot}\nabla\times\boldsymbol{b} - u_\perp\delta I_1 + \mathcal{O}(\delta^2), \qquad (31b)$$

$$\bar{\mu} = \mu - \delta\frac{v_\parallel}{B}\mu\boldsymbol{b}\boldsymbol{\cdot}\nabla\times\boldsymbol{b} + \frac{v_\parallel}{B}u_\perp\delta I_1 + \mathcal{O}(\delta^2), \qquad (31c)$$

$$\bar{\phi} = \phi + \delta I_2 + \mathcal{O}(\delta^2). \qquad (31d)$$

The contributions $I_1(\phi)$ and $I_2(\phi)$ are integrals of the form introduced in Eq. (25). These terms are small, $\mathcal{O}(\delta)$, and periodic in ϕ. The second term on the RHS of Eq. (31b) ensures that $\bar{v}_\parallel = \dot{\boldsymbol{R}}\boldsymbol{\cdot}\boldsymbol{b}$, as can be seen from the first term on the RHS of Eq. (27a). The second term on the right in Eq. (31c) matches this correction to maintain energy conservation. The difference between $\boldsymbol{R}$ and $\bar{\boldsymbol{R}}$ is $\mathcal{O}(\delta^2)$ and does not play a role in first order guiding center theory.

We take the set $(\boldsymbol{R}, \bar{v}_\parallel, \bar{\mu}, \bar{\phi})$ as our new phase space variables. The guiding center equations are

$$\dot{\boldsymbol{R}} = \bar{v}_\parallel\boldsymbol{b} + \boldsymbol{v}_E + \frac{\delta}{B}\boldsymbol{b}\times\left(\bar{\mu}\nabla B + v_\parallel^2\nabla_\parallel\boldsymbol{b}\right) + \mathcal{O}(\delta^2), \qquad (32a)$$

$$\dot{\bar{v}}_\parallel = \frac{1}{\delta}E_\parallel - \bar{\mu}\nabla_\parallel B + \bar{v}_\parallel\boldsymbol{v}_g\boldsymbol{\cdot}\nabla_\parallel\boldsymbol{b} + \mathcal{O}(\delta^2), \qquad (32b)$$

$$\dot{\bar{\mu}} = \mathcal{O}(\delta^2). \qquad (32c)$$

The equation for the new gyro-phase $\bar{\phi}$ is identical to Eq. (27d).

Equation (32c) shows that the new magnetic moment $\bar{\mu}$ is constant on the time scale on which the fields vary. It can be shown [8,9] that expression (32c) gives the first terms of a series expansion to powers of δ of the adiabatic constant, $\bar{\mu} = \mu + \cdots + \mathcal{O}(\delta^n)$, which has the property that its time derivative is of the order of the neglected terms, $d\bar{\mu}/dt = \mathcal{O}(\delta^n)$. Indeed, for the true adiabatic constant we expect changes of the order $\Delta\mu \sim e^{-\omega_c\tau}$. Since the exponent scales as $-1/\delta$, $\Delta\mu$ vanishes faster than any power δ^n for $\delta \to 0$.

According to Eqs. (31b,c) the particle energy is

$$\epsilon = \mu B + \tfrac{1}{2}v_\parallel^2 = \bar{\mu}B + \tfrac{1}{2}\bar{v}_\parallel^2 + \mathcal{O}(\delta^2).$$

Following the treatment in Sec. IV. D the distribution function in terms of the phase space variables $(\boldsymbol{R}, \epsilon, \bar{\mu}, \bar{\phi})$ is

$$f(\boldsymbol{x}, \boldsymbol{v}, t) = F(\boldsymbol{R}, \epsilon, \bar{\mu}, \bar{\phi}, t),$$

and the kinetic equation for F is

$$\Big(\frac{\partial}{\partial t} + \dot{\boldsymbol{R}}\boldsymbol{\cdot}\frac{\partial}{\partial\boldsymbol{R}} + \dot{\epsilon}\frac{\partial}{\partial\epsilon} + \dot{\bar{\mu}}\frac{\partial}{\partial\bar{\mu}} + \dot{\bar{\phi}}\frac{\partial}{\partial\bar{\phi}}\Big)F = C(F).$$

Upon substituting Eqs. (32), neglecting contributions of $\mathcal{O}(\delta)$, and averaging the resulting expression over $\bar{\phi}$, we obtain the **drift kinetic equation** for the $\bar{\phi}$-averaged distribution function $\bar{F}(\boldsymbol{R}, \epsilon, \bar{\mu}, t) \equiv \langle F\rangle$,

$$\Big[\frac{\partial}{\partial t} + \boldsymbol{v}_g\boldsymbol{\cdot}\nabla + \big(\bar{\mu}\frac{\partial B}{\partial t} + \frac{q}{m}\boldsymbol{v}_g\boldsymbol{\cdot}\boldsymbol{E}\big)\frac{\partial}{\partial\epsilon}\Big]\bar{F} = \langle C(F)\rangle.$$

Introducing the Jacobian of the transformation $(\boldsymbol{x}, \boldsymbol{v}) \to (\boldsymbol{R}, \epsilon, \bar{\mu}, \bar{\phi})$,

$$J = \frac{B(\boldsymbol{R})}{\bar{v}_\parallel} + \delta\boldsymbol{b}\boldsymbol{\cdot}\nabla\times\boldsymbol{b} + \mathcal{O}(\delta^2),$$

the guiding center velocity (32a) can be written as

$$\boldsymbol{v}_g \equiv \dot{\boldsymbol{R}} = \boldsymbol{v}_E + \frac{\boldsymbol{B} + \delta\nabla\times\bar{v}_\parallel\boldsymbol{b}}{J} + \mathcal{O}(\delta^2).$$

This expression includes the ∇B and curvature drifts but not the polarization drift, which is $\mathcal{O}(\delta^2)$. It follows that in equilibrium situations, $(\partial/\partial t, \boldsymbol{v}_E \to 0)$,

$$\nabla\boldsymbol{\cdot}J\boldsymbol{v}_g = 0.$$

This equation shows that the drift kinetic equation conserves the number of guiding centers exactly [10], not just to $\mathcal{O}(\delta^2)$. Hence, a Hamiltonian description of the guiding center trajectories in phase space can be given.

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